GRAPHICS CALCULATORS IN THE PUBLIC EXAMINATION OF CALCULUS: MISUSES AND MISCONCEPTIONS

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Graphics calculators were allowed in all tertiary entrance mathematics examinations in Western Australia for the first time in 1998. In this paper we present an analysis of students' answers to four of the examination questions in TEE Calculus and a discussion of misconceptions attributable to the technology and misuses of it.

INTRODUCTION

The National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) and the Curriculum Framework for Kindergarten to Year 12 Education in Western Australia (Curriculum Council, 1998) recommend the use of technology for mathematics. Students should "make thoughtful use of technology to enhance their mathematical work" (Curriculum Council, 1998, p. 184). Allowing the use of graphics calculators in public examinations is part of the ongoing process of accommodating and encouraging adoption of technology into the whole curriculum. Candidates for the Western Australian Tertiary Entrance Examinations (TEE) were allowed graphics calculators in all mathematics subjects for the first time in 1998. In this paper the prevalence and nature of student use of the calculators in the Calculus TEE are discussed. In particular we consider calculator use that mediated against students providing answers acceptable 'in the public domain' (Noddings, 1990).

BACKGROUND

Widespread introduction of graphics calculators occurred in West Australian schools after the Secondary Education Authority (1995), now called the Curriculum Council, announced that graphics calculators would be allowed for all 1998 mathematics tertiary entrance examinations. Graphics calculators without symbolic processing and the Hewlett Packard HP38G with limited symbolic processing were approved (Secondary Education Authority, 1996; Curriculum Council, 1998) and students were advised that they would not be required to clear the memories of their calculators before the examinations. For the three hour 1998 Calculus TEE no explicit instructions were given to use a graphics calculator. The use of graphics calculators was at the students' discretion except in two part questions that specified an analytical approach. The paper was graphics calculator 'active' (Jones & McCrae, 1996) with student access to the calculators not only allowed but assumed (Secondary Education Authority, 1996) and one question could not be solved without using the technology.

DATA COLLECTION

The Calculus examination paper comprised nineteen questions of which six were selected for analysis. The selection was guided by Kemp, Kissane, and Bradley's (1996) categories "Graphics calculators are expected to be used . . . Graphics calculators are expected to be used by some students but not by others" (p. 108). For each of the six questions, examination markers were asked by the Curriculum Council to record the methods students used and the part marks awarded for answers for a sample of scripts. Systematic sampling (Cohen & Manion, 1994) resulted in 404 (21%) out of the total 1882 scripts being selected. These were the scripts of the first two candidates listed on each normal marks recording sheet. In addition, qualitative data relevant to this paper were obtained from three sources. Firstly, 172 (9%) of the 1882 examination scripts were perused to ascertain the details of students' working. These scripts were all papers in six randomly assigned bundles of scripts, bundled for marking and assigned randomly to us as examiners and markers. Secondly, three female and three male examination candidates were interviewed in the week following the examination. The students were chosen on the basis of differing abilities, based on their internal school assessments for Calculus for the year. The paper was used as a heuristic for the interviews, which had the purpose of ascertaining students' calculator use not apparent in written answers. Thirdly, two examination markers, both Heads of Mathematics' Departments and experienced in using the calculators, were interviewed after marking had finished. They were asked about their perceptions of the form and adequacy of students' solutions. The examination paper was the focus of discussion in each interview.

ANALYSIS

In the analysis, we consider errors that can be classed as misuses of graphics calculators and misconceptions about calculator outputs. We define as misuse the use of a calculator function or procedure that leads to an answer in unacceptable form or which does not relate to the question. Misuse might arise from a student not knowing the meaning or limitations of calculator functions. In accordance with the notion of publicly acceptable mathematical knowledge (Noddings, 1990) the traditional standards for the Calculus TEE determine what are acceptable or unacceptable forms. However, a rigorous approach has been adopted here that does not necessarily reflect how students' answers were penalised in deficit.

Misconceptions can be explained in terms of constructivist theory, which assumes that knowledge is personal (van Glasersfeld, 1990): students interpret or conceptualise mathematics individually. A student may correctly enter a function into a calculator and copy exactly the screen-display of the graph but we define this to amount to a misconception if the student has not taken into account the limitations of the calculator when interpreting the display. Because misuses and misconceptions overlap, with calculator limitations explaining both, they are not always differentiated in the discussion. In addition to these errors, we consider the time-efficiency of some approaches and the adequacy of written reasoning to support graphics-calculator assisted answers.

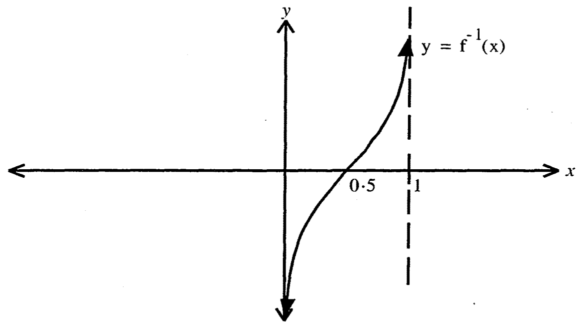
QUESTION RESPONSES

We restrict our discussion to the four questions for which errors attributable to calculator use were most widespread in the perused scripts. Each question is stated and our view of an acceptable calculator-assisted answer, informed by discussion with other markers, is given. Then, a summary of the numbers of students choosing various traditional and graphics calculator-based methods is provided, together with students' results for the various methods. The summary for each question is followed by a discussion of problematic student answers, including errors made and associated calculator usage.

For some scripts in the markers' sample, data are incomplete in that markers have identified the methods used or recorded the marks awarded for only some questions. These omissions in the recorded data are tabulated in the summaries and are accounted for by students not answering the question or using methods that were not listed as options on the data recording sheets. Calculator use associated with these methods, as suggested by the perused scripts, is discussed.

Question: Let $f(x) = \dfrac{1}{1+e^{-x}}$ for $-\infty < x < \infty$. Sketch the graph of the inverse of f, $f^{-1}(x)$, clearly indicating all intercepts and asymptotes.

Solution:



Results: The data summarised in Table 1 indicate whether students first graphed $f(x)$ and then reflected it over $y = x$, or first calculated $f^{-1}(x)$ and then graphed it. In view of the definitions of the given function and its inverse, students most likely used their graphics calculators for graphing in both methods.

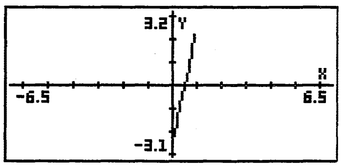
Table 1
Method Chosen by Students for Finding the Inverse Function

	Reflecting $f(x)$	Calculating $f^{-1}(x)$
No. students choosing the method	138	227
No. students whose mark was recorded	123	205
No. students with full marks	66	105
Mean mark for those with a recorded mark[1]	2.8	2.7

n = 404 (of which 39 were not recorded as choosing either of the listed methods)
[1]maximum = 4

Perusal of scripts showed that some students who chose the traditional method of calculating the inverse function $f^{-1}(x)$ sketched a graph with finite range. That is, they failed to recognise the asymptotes at $x=1$ and $x=1$. This misconception is an artifact of the screen resolution. The choice of an inappropriate scale (see Figure 1) results in a discretisation of the x-axis which does not enable a student to see the function values close to $x=1$ or $x=0$. That is, valuable information about asymptotic behaviour is not displayed and therefore lost. Another problem exhibited in students' written answers that is attributable to the calculator display (see Figure 1) was not showing the curvature of the graph, or showing it incorrectly. The mean mark of 2.7 out of a possible 4 (see Table 1) is explained by these errors, and also by students not obtaining $f^{-1}(x)$ correctly. However, 105 (51%) out of 205 students in the sample (see Table 1) who chose to calculate the inverse function scored full marks, an indication of competent calculator usage.

Figure 1
Calculator Display of $f^{-1}(x)$



Alternatives adopted by the 39 (10%) out of 404 students (see Table 1) not recorded as using the methods described above include graphing the reciprocal of $f(x)$ rather than its inverse. This is explained by students associating the graphics calculator *reciprocal* functions INVERSE and x^{-1} with the operation of finding the *inverse* of a function.

Question: Find the area of the region bounded by the curves $y = 3 + 2x - x^2$ and $y = -5$.

Solution: The area of the region is given by $\int_{-2}^{4} (3 + 2x - x^2 + 5)dx = 36$.

Results: The data summarised in Table 2 indicate whether students worked the integral symbolically, or gave only a numeric answer that suggests calculator evaluation.

Table 2
Method Chosen by Students for Evaluation of Integral

	Symbolic	Numeric Answer Only
No. students choosing the method	229	163
No. students whose mark was recorded	200	148
No. students with full marks	141	117
Mean mark for those with a recorded mark[1]	1.5	1.7

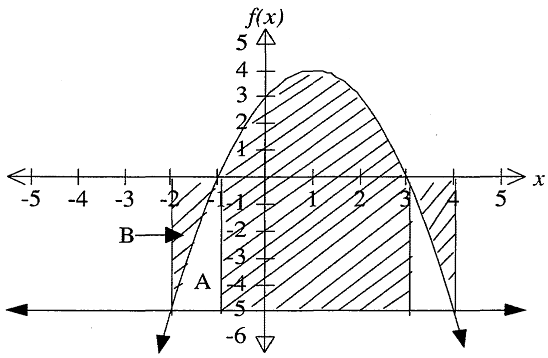
n = 404 (of which 12 were not recorded as choosing either of the listed methods)
[1]maximum = 2 for the integral, after the points of intersection had been obtained.

A number of students split up the region, possibly because coordinates of graphs are readily available on graphics calculators. This approach often led to incorrect answers because of a misidentification of the integrals representing the desired part regions. For example,

$$\int_{-2}^{-1} (3 + 2x - x^2 + 5)dx$$

represents the area of region A (see Figure 2) but some students mistakenly evaluated area of region B, and duplicated their error for the integral between $x = 3$ and 4 (see Figure 2).

Figure 2
Incorrect Representation of the Region Enclosed by $y = 3 + 2x - x^2$ and $y = -5$

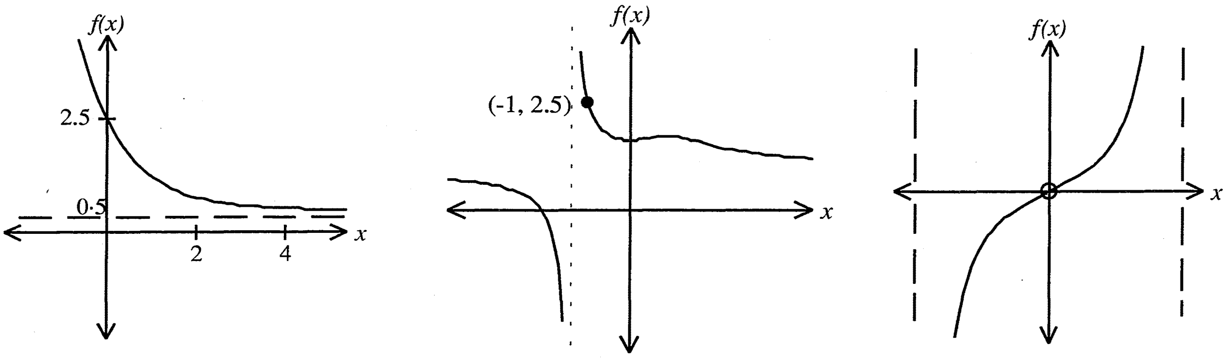


Another error was to write the answer as 35.9999, students not recognising the need to correct the inaccuracy of their calculator. The integral was relatively easy to evaluate symbolically so that the 163 (40%) out of 404 students in the sample (see Table 2) who used their calculators possibly gained no time advantage and the mean marks for the different methods were similar.

Question: Determine the following limits showing your reasoning.

(a) $\lim_{x \to \infty} \frac{e^x + 4}{2e^x}$ (b) $\lim_{x \to -1} \frac{x^3 + x^2 + 5}{x^3 + 3}$ (c) $\lim_{t \to 0} \frac{\tan^2(3t)}{t}$

Solutions:



(a) $\lim_{x \to \infty} \frac{e^x + 4}{2e^x} = 0.5$ (b) $\lim_{x \to -1} \frac{x^3 + x^2 + 5}{x^3 + 3} = 2.5$ (c) $\lim_{t \to 0} \frac{\tan^2(3t)}{t} = 0$

Results: The data summarised in Table 3 indicate whether students used graphical or numeric methods that are both potentially graphics-calculator based, or traditional symbolic methods.

Table 3
Type of Method Chosen by Students for Evaluating Limits in Question 4(a), 4(b) and 4(c)

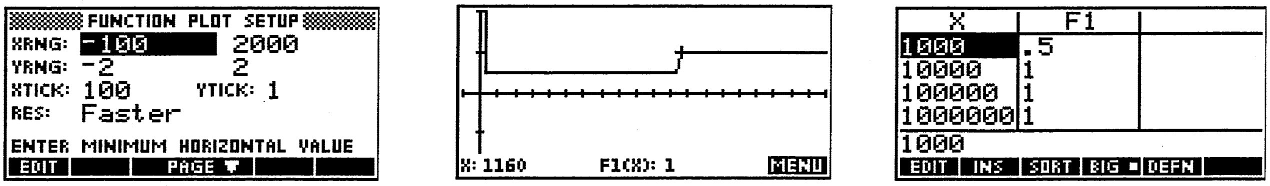
	graphical			numeric			symbolic		
	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
No. students choosing the method.	26	18	28	67	182	77	290	193	244
No. students whose mark was recorded	24	17	23	60	150	71	265	191	222
No. students with full marks	9	5	3	10	106	22	166	147	97
Mean mark for those with a recorded mark[1]	1.2	1.1	1.3	0.9	1.6	1.6	1.4	1.7	1.7

n=404 (of which 21 in (a), 11 in (b), 55 in (c) were not recorded as choosing any of the listed methods)
[1] maximum for (a) is 2, for (b) is 2, for (c) is 3

Only a small number of students (see Table 3) chose to provide graphical reasoning for the limits. For those who provided graphs, omissions were not including the asymptote in part (a), failing to identify the point (-1,2.5) either with its co-ordinates or with the scales on the axes in part (b) and not showing the discontinuity at $t = 0$ in part (c). The discontinuity is not easily apparent on a calculator-generated graph, leading to the misconception of portraying the function as continuous at the origin. Substitution reasoning was more commonly adopted than graphing (see Table 3) and often students' answers were displayed in tables similar to those on graphics calculators, suggesting that the technology was used rather than a scientific calculator. Insufficient reasoning was, in (a), providing only two substitutions which are not enough to show a trend; in (b), evaluating the expression at $x = -1$ without justification; and in (c) giving only one substitution either side of zero. The mean marks achieved for all limits for graphical and substitution approaches, and the proportions of students having chosen one of these approaches and receiving full marks, were lower than for traditional symbolic approaches (see Table 3). However, some of the differences were only marginal and for part (b) we question the large number of students recorded as using symbolic reasoning because perusal of scripts showed few instances of it.

Some students gave an answer of one for the limit in (a). This error is caused by the limitations to storing very large or very small numbers in graphics calculators. For example, for every $x > 1151.3$, e^x exceeds the capacity of the Hewlett Packard HP38G resulting in the graphical and calculation effects illustrated in Figure 3.

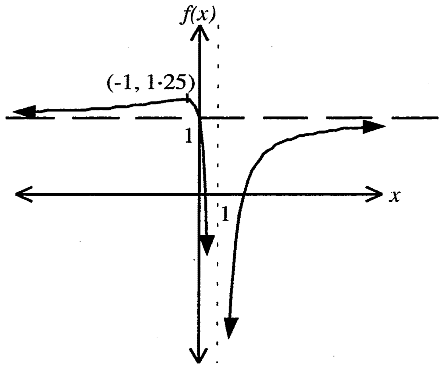
Figure 3
Graph and Table of Values for $f(x) = (e^x + 4)/(2e^x)$



Question: A function is defined by the equation $f(x) = 1 - \frac{x}{(x-1)^2}$.

Sketch the graph of $f(x)$, indicating all asymptotes and the co-ordinates of any turning points.

Solution:



Results: The data summarised in Table 4 are the results for students who chose a symbolic approach to establishing the nature of the graph, and for those who appeared to use their graphics calculator by only providing the graph.

Table 4
Approach Students Chose for Graphing f(x)

	Working Shown	Graph Only
No. students choosing the method	63	321
No. students whose mark was recorded	59	289
No. students with full marks	18	66
Mean mark for those with a recorded mark[1]	3.6	3.7

n = 404 (of which 20 were not recorded as choosing either of the listed methods)
[1]maximum = 5

The 321 (79%) out of 404 students (see Table 4) who appeared to use their graphics calculators by providing the graph without working chose the time efficient option for answering this question. However, the mean mark of 3.7 and that only 66 (23%) out of 289 students providing the graph only (see Table 4) scored full marks, indicates students encountered difficulties in interpreting the calculator screen display. Errors were the inclusion of a turning point at $x = 1$, possibly found by running the cursor along to the bottom of the graph on the calculator screen; failure to identify the turning point at $x = -1$; and for the left branch of the graph to drop below the horizontal asymptote. In fact, close to half the students whose scripts were sighted failed to include the horizontal asymptote. However, judging by the mean marks (see Table 4), students who chose to provide the graph only did not score significantly differently than those who used symbolic working.

DISCUSSION

Interviews with students as to how they used their calculators were a vital part of explaining the source of students' errors, with calculator processes not being apparent from examination scripts. One student reminded us of the misuse of the INVERSE function, which perpetuates the error of plotting the reciprocal function $y=(f(x))^{-1}$ in answering a question asking for the inverse function $y=f^{-1}(x)$. Another gave a rich description of the difficulties encountered while attempting to interpret the screen-display of a graph. The analysis indicates that interpretation and transcription of graphs are major areas of difficulty for many students. In summary, errors were to interpret and copy graphical screen-displays so that (a) a function stopped on vertical asymptotes instead of approaching them; (b) a point discontinuity was not located; (c) the limiting value of a function was believed to be correct even though the capacity of the calculator to store large numbers had been exceeded; (d) horizontal asymptotes were omitted, and a function drawn to drop below its limiting value; (e) a non-existent turning point was located on an asymptote; and (f) a turning point was not located even when the question suggested one existed. Boers and Jones (1994) observed students to have similar difficulties with point discontinuities and asymptotes and Tobin (1995) discusses students' difficulties in copying calculator-generated graphs like we noted.

Other errors were incorrectly subdividing the integral associated with finding an area between two curves, and failing to round answers that were generated inaccurately on the calculator. In addition, students omitted to provide sufficient examples to support calculator-assisted numeric evaluation of limits, and to adequately label graphs when using them as reasoning for evaluating limits.

The outcomes for the last question discussed above indicate that a significant number of students (21%, n = 404) did not use their calculator for graphing even when there was a time-advantage to do so. This might suggest underutilisation of the technology as the first option for solving problems by some students, but does not preclude them from having used their calculators for checking. Underutilisation of the technology can also arise from students not knowing all the functions of their calculator appropriate for a subject, which was the case with some of the interviewed students, and from misinterpreting 'show reasoning' (see limit question) to mean 'show algebraic reasoning'. The reverse problem in questions not discussed here was of students providing graphical or numeric reasoning when an analytical approach was specified.

The results reported here suggest that calculator-based answers are not associated, in general, with higher (or lower) marks than traditional alternatives. The value of graphics calculators might therefore primarily be to develop students understanding in the teaching/learning process rather than as a tool to maximise assessment scores.

CONCLUSION

The first Calculus TEE where graphics calculators have been allowed has highlighted problem areas deriving from use of the technology. Of these the foremost are the interpretation of graphical information and students' apparent uncertainty as to when use of graphics calculators is appropriate. In this first examination with the technology, this uncertainty was accompanied by students' underutilisation of the calculators, which may cease to be a problem as calculator familiarity increases. Problems with the interpretation of graphical information indicate that instructional time needs to be spent on exploring the limitations of the calculators.

We sincerely thank the students and markers who contributed to our inquiry, and to the Curriculum Council of Western Australia for their support. Comments in this paper are not to be taken to represent the views of the Curriculum Council.

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